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1989 J. Phys. A: Math. Gen. 22 L763

(http://iopscience.iop.org/0305-4470/22/16/001)

J. Phys. A: Math. Gen. 22 (1989) L763-L767. Printed in the UK

LETTER TO THE EDITOR

Action and kinematical integral geometry

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Received 3 May 1989

Abstract. An interpretation of the classical non-relativistic or relativistic action for a point free particle is given in terms of Galilean or Minkowskian integral geometry. That interpretation remains valid for some interactions.

As is well known, the action for a point particle in relativity is proportional to the length of its worldline, and therefore has a direct geometrical meaning. That is not the case for the non-relativistic mechanics, where the worldline length is the (universal) time, which is path independent, whereas action does not have any known geometrical interpretation.

In plane Euclidean geometry there are some relationships between the length of any (open) arc of curve and the measure of the subset of all straight lines which intersect this arc; their study belongs to integral geometry [1]. A naive attempt to find an analogous result for curves in classical or relativistic spacetimes gives divergences in the integrals, due to the non-compact nature of the 'rotation' (i.e. boost) subgroups. We report here some results of a more refined attempt [2] which allows one to associate to every *closed* loop (i.e. two arcs of time-like curves with the same endpoints) an integral over the set of all (time-like) lines of the number of *oriented* intersections with the loop; this quantity is free of divergences, and surprisingly, its value turns out to have an unexpected meaning: it is equal to the *difference* of actions of a free particle along the two paths, both in relativistic and non-relativistic mechanics.

These results are very satisfactory because they show that a purely geometrical analysis points not to the action for an open path (which is not gauge invariant), but to the action along a closed path which *is* the gauge-invariant relevant quantity in quantum mechanics, as shown in Feynman's formulation.

Finally we show that the aforementioned relationship does not only hold for the free case but also for linear and quadratic potentials. In another direction, it is possible to extend these results to all plane Cayley-Klein geometries, where the connection with the generalised Gauss-Bonnet theorem is clearly apparent. A detailed exposition will be the subject of two forthcoming publications [2, 3]. Some aspects of plane Cayley-Klein geometries can be found in [4-7].

The length L of a line Γ in the Euclidean plane appears in the context of integral geometry [1] in the following connection. If Γ is a piecewise differentiable (closed or not) curve, and $N_\Gamma(l)$ denotes the ordinary number of intersections with Γ of a generic straight line l with cartesian equation $x \cos\theta + y \sin\theta = p$, we have the Cauchy-Crofton formula

$$\int N_\Gamma(l)\, \mathrm{d}l = 2L_\Gamma \tag{1a}$$

0305-4470/89/160763+05$02.50

where $dl = dp \wedge d\theta$ is a 2-form in the set of unoriented lines, determined (up to a factor) by the condition of being invariant under the Euclidean group. A further variation of this formula can be obtained by splitting a closed curve into two arcs Γ_1 and Γ_2 which are considered as two oriented space paths from A to B or as a loop based on A and made up of two arcs Γ_1 and $-\Gamma_2$, with different orientations. If we define an 'oriented total intersection number' of a straight line l with (Γ_1, Γ_2) as the number of intersections of l with Γ, *minus* the number of intersections with Γ_2, we obtain

$$\int N_\Gamma(l)\, dl = 2(L_1 - L_2). \tag{1b}$$

We only recall that $(1a)$ is a particular instance [1] of the formula

$$\int_{L_r \cap M^q \neq \varnothing} N(L_r \cap M^q)\, dL_r = \frac{O_n \ldots O_{n-r+1}}{O_r \ldots O_1} \sigma_q(M^q) \tag{2}$$

for compact q-dimensional manifolds in n-dimensional Euclidean space, O_n being the measure of the n-dimensional sphere.

Consider now a motion in non-relativistic spacetime given by a time-like curve Γ and for the sake of simplicity let us take one space dimension only. It is almost evident that $(1a)$ does not have any sensible analogue, the difficulty being traced back to the non-compact nature of the subgroup of Galilean boosts, which makes divergent the integral in the RHS of $(1a)$ (with an invariant measure $dl = dk \wedge ds$ in the set of time-like straight lines $x = kt + s$). This is also the case, *mutatis mutandis*, in Minkowski spacetime. There is, however, a new feature in these two cases owing to the fact that a (time) orientation can be invariantly assigned to time-like lines. A relative orientation $\pm$ can be invariantly assigned to each intersection of two time-like lines, according to whether the orientation of these lines is the same or opposite. If now we sum over all the intersections of Γ with l, counting each positive intersection as 1 and each negative intersection as -1, we obtain an 'oriented total intersection number' $(N_\Gamma(l)$. Consider two points A and B in spacetime and Γ_1, Γ_2 two future time-like curves from A to B, given, say, by

$$t \in [t_A, t_B] \to (t, x_i(t)) \qquad \text{with} \begin{cases} x_1(t_A) = x_2(t_A) \\ x_1(t_B) = x_2(t_B). \end{cases} \tag{3}$$

The pair (Γ_1, Γ_2) can be considered as a closed (piecewise differentiable) time-like loop, which is made up of a future arc from A to B and a past one from B to A with the opposite orientation. The interesting point is that the integral in $(1b)$ with the *oriented* total intersection number N_Γ is free of divergences, and defines a geometric quantity associated with the loop. This happens both in the non-relativistic as well in the relativistic case. But in the non-relativistic case the RHS of $(1b)$ cannot surely be the difference of geometric lengths of the arcs Γ_1 and Γ_2, which is identically equal to zero (universal time). What is the meaning of this quantity associated with the loop? The somewhat surprising answer is the following.

Theorem 1. Let Γ_i, $i = 1, 2$, be the two differentiable future time-like curves in the non-relativistic $(1+1)$-dimensional spacetime given by (3). If $N_\Gamma(k, s)$ denotes the total oriented number of intersections of the future time-like straight line (k, s) with $\Gamma \equiv (\Gamma_1, \Gamma_2)$, the following integral relation holds:

$$\int N_\Gamma(k, s)\, dk \wedge ds = \int_{t_A}^{t_B} [(\dot{x}_1(t))^2 - (x_2(t))^2]\, dt. \tag{4}$$

Hence the integral of the total oriented intersection number over the set of all future time-like straight lines equals (twice) the *difference* of the actions (per unit mass) of a free particle going from A to B through Γ_1 and Γ_2. The same connection does also happen in the relativistic case, where the result is less appealing because the action is defined there as proportional to the path length, and (4) is but an extension to this case of $(1a)$.

These results can be linked to the Gauss–Bonnet theorem for the set of all future time-like straight lines. Area S and angular excess $\alpha+\beta-\gamma$ of a triangle (α, β internal smaller angles, γ the external larger one) are related in any two-dimensional Riemannian space of non-zero constant curvature K by

$$K|S| = \alpha+\beta-\gamma \tag{5}$$

but that relation degenerates into an identity for $K=0$, leaving the area S to be independent of the (identically equal to zero) angular excess. Written in this way, (5) holds for the nine plane Cayley-Klein geometries, which includes the geometries of the set of all lines (time-like when needed) in the Euclidean, Galilean or Minkowskian planes (named with the prefix co-; see [8] for a specific discussion of the Minkowskian case). The easiest way to see the relation between (4) and (5) is to take for Γ, a segment of future straight line from A to B, and let Γ_2 be a curve made up of two future straight segments from A to C and from C to B. Let a, b, c be the lengths of the segments CB, AC, AB; so that $L_1 = c$, $L_2 = a+b$ and $L_1 - L_2 = a+b-c$. Hence the triangle ABC determines a dual triangle in the set of all future time-like straight lines and the absolute value of the integral in the RHS of $(1b)$ can be seen to be equal to twice the area $|S^*|$ of that dual triangle; also by duality the angular excess of the dual triangle is equal to $a+b-c$. So we have an equation

$$K^*|S^*| = a+b-c \tag{6}$$

with $K^* = 1, 0, -1$ for the Euclidean, Galilean and Minkowskian cases respectively (the constant curvature of co-Euclidean, co-Galilean and co-Minkowskian planes). As a consequence, we recognise why in the Euclidean and Minkowskian cases $\int N_\Gamma(l)\,\mathrm{d}l$ appears as twice the difference of lengths, but is independent of the (identically equal to zero) difference between the lengths of the two paths for the Galilean case.

For the $(2+1)$-dimensional and $(3+1)$-dimensional cases, theorem 1 has a straightforward generalisation. We give only the statement.

Theorem 2. Let Γ_i, $i=1,2$, be the two differentiable future time-like curves in the Galilean $(3+1)$-dimensional spacetime, as in (3). If $N_\Gamma(k,s)$ denotes the total oriented number of intersections of the inertial 3-plane l_3 with $\Gamma \equiv (\Gamma_1, \Gamma_2)$, the following integral relation holds:

$$\int N_\Gamma(l_3)\,dl_3 = \frac{8\pi}{3}\int_{t_A}^{t_B} [\tfrac{1}{2}(\dot{x}_1(t))^2 - \tfrac{1}{2}(\dot{x}_2(t))^2]\,\mathrm{d}t. \tag{7}$$

The above results hold for a free particle. Can these results be extended to the case of a particle interacting with an external field? For this case the natural candidates for 'straight lines' are the 'true' motions of the particle under this field. The main difficulty is then to define a density for the set of the 'physical' motions. In the free case, the kinematical group (Galilei, Poincaré) is the symmetry group of motions, and the requirement of invariance under this group determines (up to a factor) the density.

But the presence of a potential breaks the kinematical symmetry. Returning for a moment to the geometrical context, formula (1a) also holds in a surface of non-zero curvature, with an adequate density for geodesics. Therefore one can hope that equations (4) and (7) can be extended to cover the motion of a point particle in an external potential, where now the RHS are replaced by the difference of actions with the potential terms. Whereas a complete proof of this conjecture is not available, it is possible to show explicitly that the results of theorems 1 and 2 (insofar as they say that 'the measure of geometric elements intersecting Γ' is proportional to 'the difference of actions along Γ_1 and Γ_2') are also valid for time-independent polynomial potentials of degree one or two.

We are going to develop here the oscillator one-dimensional case. The potential is $V(x)=\frac{1}{2}(\omega^2x^2)$ and a physical motion given by

$$x(t)=(s/\omega)\cos(\omega t)+(k/\omega)\sin(\omega t) \tag{9}$$

can be parametrised by (k, s). Define a density $\mathrm{d}l=\mathrm{d}k\wedge\mathrm{d}s$ for these motions. Let (Γ_1,Γ_2) be a time-like curve where A and B are the common endpoints of the curves Γ_1 and Γ_2. A line (k, s) intersects Γ_i at the point $(t, x_i(t))$ if and only if $s\cos(\omega t)=\omega x_i(t)-k\sin(\omega t)$. This equation determines s as a function of k, except for a zero measure set of values of t, which is irrelevant when integrating. Then

$$s_i(t)=(\omega x_i(t))/\cos(\omega t)-k\tan(\omega t). \tag{10}$$

This is the equation of a line in the (k, s) plane. When t varies, the motion of this line is a rotation with centre given by the solution of $\mathrm{d}s_i(t)/\mathrm{d}t=0$. The computation of the integral in the LHS of (1b) is made in two steps. First the integral over all lines of the number of intersections of a generic line (k, s) with Γ_1 is:

$$H_1=\int N_\Gamma(l)\,\mathrm{d}l=\int_{t_A}^{t_B}\mathrm{d}t\left(\int_{-\infty}^{k_1(t)}\frac{\mathrm{d}s_1(t)}{\mathrm{d}t}\,\mathrm{d}k-\int_{k_1(t)}^{+\infty}\frac{\mathrm{d}s_1(t)}{\mathrm{d}t}\,\mathrm{d}k\right). \tag{11}$$

A similar expression is obtained for Γ_2. Then the number of the total oriented intersections with Γ is $H=H_1-H_2$:

$$\begin{aligned}H&=\int_{t_A}^{t_B}\mathrm{d}t\left(\int_{-\infty}^{k_1(t)}\frac{\mathrm{d}s_1(t)}{\mathrm{d}t}\,\mathrm{d}k-\int_{k_1(t)}^{+\infty}\frac{\mathrm{d}s_1(t)}{\mathrm{d}t}\,\mathrm{d}k-\int_{-\infty}^{k_2(t)}\frac{\mathrm{d}s_2(t)}{\mathrm{d}t}\,\mathrm{d}k+\int_{k_2(t)}^{+\infty}\frac{\mathrm{d}s_2(t)}{\mathrm{d}t}\,\mathrm{d}k\right)\\&=\int_{t_A}^{t_B}\mathrm{d}t\left(-\int_{k_2(t)}^{-k_1(t)}\frac{\mathrm{d}(s_1(t)-s_2(t))}{\mathrm{d}t}\,\mathrm{d}k+\int_{k_2(t)}^{k_1(t)}\frac{d(s_1(t)+s_2(t))}{\mathrm{d}t}\,\mathrm{d}k\right).\end{aligned} \tag{12}$$

With $s(t)$ given by (10), $\mathrm{d}s_i(t)=0$ implies that

$$k_i(t)=(\mathrm{d}x_i(t)/\mathrm{d}t)\cos(\omega t)+x_i(t)\omega\sin(\omega t). \tag{13}$$

Then (12) gives after a short computation

$$H=\int_{t_A}^{t_B}\{[(\dot{x}_1)^2-(\dot{x}_2)^2]+\omega^2[(x_1)^2-(x_2)^2]\tan^2(\omega t)+2\omega(\dot{x}_1x_1-\dot{x}_2x_2)\tan(\omega t)\}\,\mathrm{d}t.$$

An integration by parts using the boundary conditions finally gives

$$\int N_\Gamma(l)\,\mathrm{d}l=2\int_{t_A}^{t_B}\{[\tfrac{1}{2}(\dot{x}_1(t))^2-\tfrac{1}{2}\omega^2(x_1(t))^2]-[\tfrac{1}{2}(\dot{x}_2(t))^2-\omega^2(x_1(t))^2]\}\,\mathrm{d}t \tag{14}$$

which is the announced result. Why is the choice $\mathrm{d}l=\mathrm{d}k\wedge\mathrm{d}s$ the correct one? The answer lies in the additional symmetry of the oscillator: the Euclidean structure of the set of all oscillator motions. It is well known that the harmonic oscillator can be considered as a free particle in an oscillating Newton–Hooke universe, i.e. in a spacetime manifold whose kinematical group is the Newton–Hooke group. The homogeneous subgroup is isomorphic to the Euclidean group, which acts on the set of motions in the standard form, the coordinates s and k being cartesian coordinates on this plane and if we require a density invariant under the complete Euclidean group, the only choice is $\mathrm{d}l = \mathrm{d}k \wedge \mathrm{d}s$.

For the case of a particle in an homogeneous force field a similar result can be obtained, because the set of motions can be also identified with an homogeneous space of the Galilei group, and a density can be singled out applying the same ideas. Therefore we have the following result.

Theorem 3. For a particle in a homogeneous force field or an harmonic oscillator, a density $\mathrm{d}l$ can be defined in the set of all physical motions in such a way that the integral of the total number of intersections with (Γ_1, Γ_2) over the set of physical motions is equal to the difference between the actions for a particle moving along Γ_1 and Γ_2, with the corresponding potential $V(x)$, i.e.

$$\int N_\Gamma(l)\,\mathrm{d}l = 2\int_{t_\mathrm{A}}^{t_\mathrm{B}} \{[\tfrac{1}{2}(\dot{x}_1(t))^2 - V(x_1(t))] - [\tfrac{1}{2}(\dot{x}_2(t))^2 - V(x_2(t))]\}\,\mathrm{d}t. \tag{15}$$

We thank the Comisión Interministerial de Ciencia y Technologia of Spain (CICYT) and the Caja de Ahorros de Salamanca for partial financial support.

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